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Non-standard deformation of B_n series

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Abstract. A generalization of the quantized enveloping algebra of B_n ($U_q(B_n)$) is constructed. This algebra which we call $X_q(B_n)$, is based on a non-standard R-matrix corresponding to the series B_n, and yields $U_q(B_n)$ in a special case.

1. Introduction

Since the advent of the q-deformation of the universal enveloping algebras of the series A_n, B_n, C_n and D_n, [1,2], there has been a great deal of activity on constructing new examples of quantum groups [3–5]. A partial study of the quantum groups associated with the non-standard R matrices of the series B_n, D_n and C_n has been reported in [4]. We say a partial study since the relations derived in [4] are not sufficient to characterize an algebra, in particular an analogue of the Poincare–Birkhof–Wit basis can not be built by using these relations. The most important property of these Hopf algebras is the nil-potency of some of the generators which brings about many particularities in their representation theory [3]. In some cases they have also been related to Ribbon–Hopf algebras [5] which in turn are important for constructing invariants of Ribbon graphs [6] and 3-manifolds. In [7] we have studied in detail the non-standard quantum group associated with the exotic R-matrix corresponding to the series A_{n-1}. In this paper we continue our study for the series B_n and construct a generalization of the quantized universal enveloping algebra $U_q(B_n)$, which we call $X_q(B_n)$. This new Hopf algebra has interesting new features among which is the nil-potency of certain elements. One of the urgent problems concerning these quantum groups is their relations with superalgebras. This problem has been addressed only for the simple case of $X_q(A_1)$ [5].

2. The structure of $U_q(B_n)$

The R-matrix for the B_n series is

$$R=\sum_{i}^{N} e_{ii}\otimes e_{ii}+\sum_{i\neq j,j'}^{N} e_{ii}\otimes e_{jj}+\sum_{i>j}^{N}(q-q^{-1})e_{ij}\otimes e_{ji}$$
$$+q^{-1}\sum_{i\neq i'}^{N} e_{i'i'}\otimes e_{ii}-(q-q^{-1})\sum_{i>j}^{N} q^{\rho_i-\rho_j}e_{ij}\otimes e_{i'j'} \qquad (1)$$

where

$$N = 2n+1 \qquad i' = N+1-i$$
$$(\rho_1, \rho_2, \ldots, \rho_N) = \left(n-\tfrac{1}{2}, n-\tfrac{3}{2}, \ldots, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \ldots, -n+\tfrac{1}{2}\right).$$

The above R-matrix has an important property,

$$R = C_1(R^{t_1})^{-1}C_1^{-1} = C_2(R^{t_2})^{-1}C_2{}^{-1} \tag{2}$$

where t_1 and t_2 stand for the transposition with respect to the first and second factors respectively and

$$C = C_0 q^{\rho} \qquad \rho = \mathrm{diag}(\rho_1, \ldots, \rho_N)$$

and C_0 is the diagonal matrix with matrix elements $(C_0)_{ij} = \delta_{ij'}$. Using (2) it can be shown that $X^t CX$ plays the role of invariant quadratic form, where X_i are the coordinates of the quantum plane.

The quantum matrix algebra for the orthogonal group of type B_n, $\mathcal{A}_q$ is defined by the relations

$$RT_1T_2 = T_2T_1R. \tag{3}$$

Because of the symmetry (2), there are additional relations [2]

$$TCT^tC^{-1} = CT^tC^{-1}T = 1. \tag{4}$$

The algebra of functions on the quantum group of type B_n is the quotient algebra of the $\mathcal{A}_q$ by the relations (4). Hopf algebra $U_q(B_n)$ is defined as a subalgebra of the dual algebra to $\mathcal{A}_q$, which is defined by the following evaluations

$$\langle L^{\pm}, T_1T_2\ldots T_k\rangle = R_1^{\pm}\ldots R_k^{\pm} \qquad R^+ = PRP \qquad R^- = R^{-1}. \tag{5}$$

This leads to

$$RL_2^{\pm}L_1^{\pm} = L_1^{\pm}L_2^{\pm}R \tag{6}$$
$$RL_2^{+}L_1^{-} = L_1^{-}L_2^{+}R. \tag{7}$$

Using $\det R = 1$ and the above duality [2] one arrives at

$$L_{11}^{+}\ldots L_{NN}^{+} = 1 \qquad L_{ii}^{+}L_{ii}^{-} = L_{ii}^{-}L_{ii}^{+} = 1 \tag{8}$$

and symmetry of R leads to

$$L^{\pm}C^t(L^{\pm})^t(C^{-1})t = C^t(L^{\pm})^t(C^{-1})^tL^{\pm} = 1. \tag{9}$$

Using (8) and (9) $L_{n+1,n+1}^{\pm}$ become identity and L_{ii}^{+} can be identified with $(L_{ii}^{-})^{-1}$ and with $(L_{i'i'}^{+})^{-1}$. The lines $L_{ii'}^{\pm}$ do not correspond to the root system and in the classical limit $q \to 1$ are dropped out of the commutation relations (6). The monodromy matrices L^+ and L^- accommodate the q-analogues of the Cartan–Weyl basis. Since the structure of $U_q(B_n)$ can be defined in the Chevalley basis, it is sufficient to identify only those elements of the monodromy matrices which correspond to the simple roots. As an example we consider the simple case $U_q(B_2)$. The generalization is completely straightforward. In order to avoid the necessity of any further redifinition of generators, from the outset, we define the monodromy matrices $L\pm$ as follows:

$$L^+ = \begin{pmatrix} k_1 & w(k_1k_2)^{\frac{1}{2}}X_1^+ & \beta k_1{}^{\frac{1}{2}}E_1 & \gamma k_1^{\frac{1}{2}}E_2k_2^{-\frac{1}{2}} & O_1 \\ & k_2 & \beta k_2{}^{\frac{1}{2}}X_2^+ & O_2 & -\gamma k_2^{\frac{1}{2}}E_2'k_1{}^{-\frac{1}{2}} \\ & & 1 & -\beta X_2^+k_2{}^{-\frac{1}{2}} & -\beta E_1'k_1{}^{-\frac{1}{2}} \\ & & & k_2{}^{-1} & -wX_1^+(k_1k_2)^{-\frac{1}{2}} \\ & & & & k_1^{-1} \end{pmatrix} \tag{10}$$

$$L^- = \begin{pmatrix} k_1^{-1} & & & & \\ -w(k_1k_2)^{-\frac{1}{2}}X_1^- & k_2^{-1} & & & \\ -\beta k_1^{-\frac{1}{2}}F_1 & -\beta k_2^{-\frac{1}{2}}X_2^- & 1 & & \\ -\gamma k_1^{-\frac{1}{2}}F_2k_2^{\frac{1}{2}} & O_2^- & \beta X_2^-k_2^{\frac{1}{2}} & k_2 & \\ O_1^- & \gamma k_2^{-\frac{1}{2}}F_2'k_1^{\frac{1}{2}} & \beta F_1'(k_1)^{\frac{1}{2}} & wX_1^-(k_1k_2)^{\frac{1}{2}} & k_1 \end{pmatrix} \tag{11}$$

where $w = q - q^{-1}$, $\gamma = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ $\beta = (w\gamma)^{\frac{1}{2}}$.

In this section we give only the most important commutation relations obtained from (6), (7). These are the relations which specify the form of the algebra in the Chevalley basis. Solution of (6) gives

$$k_iX_i^+ = q^{-1}X_i^+k_i \qquad k_2X_1^+ = qX_1^+k_2 \qquad k_1X_2^+ = X_2^+k_1 \tag{12}$$

$$k_iE_i = q^{-1}E_ik_i \qquad k_2E_1 = E_1k_2 \qquad k_1E_2 = q^{-1}E_2k_1 \tag{13}$$

$$[X_1^+, X_1^-] = \frac{k_2k_1{}^{-1} - k_1k_2{}^{-1}}{q - q^{-1}} \tag{14}$$

$$[X_2^+, X_2^-] = \frac{k_2^{-1} - k_2}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \tag{15}$$

$$[X_1^+, X_2^+]_q = -E_1 \tag{16}$$

$$[X_1^+, E_1]_{q^{-1}} = 0 \qquad [X_2^+, E_1] = -E_2 \tag{17}$$

$$[X_1^+, E_2]_{q^{-1}} = 0 \qquad [X_2^+, E_2]_q = 0 \tag{18}$$

where, $[a, b]_q = q^{\frac{1}{2}}ab - q^{-\frac{1}{2}}ba$. Identifying k_2/k_1 with q^{H_1} and k_2^{-1} with $q^{H_2/2}$, one obtains $U_q(B_2)$ as given by Drinfeld and Jimbo [1]

$$[H_i, X_j^{\pm}] = \pm a_{ij}X_j^{\pm} \tag{19}$$

$$[X_i^+, X_j^-] = \delta_{ij}\frac{q^{d_iH_i} - q^{-d_iH_i}}{q^{d_i} - q^{-d_i}} \tag{20}$$

where $d_i = \mathrm{diag}(1, \frac{1}{2})$ is the symmetrizer of the Cartan matrix $a_{ij} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$ is the Cartan matrix of B_2.

The Serré relations are obtained by eliminating E_1 and E_2 from (16)–(18). They read,

$$X_1^{+2}X_2^+ - (q+q^{-1})X_1^+X_2^+X_1^+ + X_2^+X_1^{+2} = 0 \tag{21}$$

$$X_2^{+3}X_1^+ - (1+q+q^{-1})(X_2^{+2}X_1^+X_2^+ - X_2^+X_1^+X_2^{+2}) - X_1^+X_2^{+3} = 0\,. \tag{22}$$

From the standard co-actions of FRT [2] the co-product of B_2 can be obtained

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{23}$$

$$\Delta(X_i^+) = q^{-\frac{1}{2}d_iH_i} \otimes X_i^+ + X_i + \otimes q^{\frac{1}{2}d_iH_i}\,. \tag{24}$$

The following relations are also obtained from (6)

$$[X_1^+, X_2^+]_{q^{-1}} = -E_1' \tag{25}$$

$$[X_1^+, E_1']_q = 0 \qquad [X_2^+, E_1'] = -E_2' \tag{26}$$

$$[X_1^+, E_2']_q = 0 \qquad [X_2^+, E_2']_{q-1} = 0\,. \tag{27}$$

This is an example of a general feature in the embedding of positive roots of the quantum algebras in the series B_n, C_n and D_n in the L^+ matrices. While the elements L_{ij}^+ above the forbidden line are obtained via q-adjoint action of simple roots, their images with respect to this line are obtained via (q^{-1})-adjoint action. Thus there are two copies of roots in monodromy matrices.

3. The non-standard case

In this section we want to discuss the non-standard case. The following generalization of the R-matrix corresponding to the series B_n was first obtained in [4]:

$$R = \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j, j'} e_{ii} \otimes e_{jj} + \sum_{i>j} w e_{ij} \otimes e_{ji} + \sum_{i \neq i'} q_i^{-1} e_{i'i'} \otimes e_{ii} - w \sum_{i>j} r(j) r(i') e_{ij} \otimes e_{i'j'} \tag{28}$$

where $q_i = q$ or $-q^{-1}$, $q_k = q_{k'}$, $q_{n+1} = 1$, $w = (q - q^{-1})$ and the coefficents $r(i)$ are

$$r(i) = \begin{cases} q_i^{-\frac{1}{2}} \prod_{j=n+1}^{i} q_j & i \geqslant n+1 \\ r^{-1}(i') & i < n+1\,. \end{cases} \tag{29}$$

Here we have found it convenient to use a different labelling than that of [4] for the rows and columns of the R-matrix. The most important feature of this R-matrix is that the parameters q_i, can be either q or $-q^{-1}$. When all the parameters q_i are set equal to q, the standard R-matrix of B_n is recovered. We call the algebra generated by $L_{ij}^{\pm}$, $X_q(B_n)$. Since in the previous sections we have given a detailed analysis of the $U_q(B_2)$ and have also remarked how the root system and its commutation relations can be obtained from the FRT formalism, we study only the most important features that arise in the non-standard case of B_n. Take the monodromy matrices as follows:

$$L^+ = \sum_{i=1}^{2n+1} k_i e_{ii} + \sum_{i=1}^{2n} \alpha_i (k_i k_{i+1})^{\frac{1}{2}} X_i^+ e_{ii+1} + \sum_{i<j-1} \alpha_{ij} (k_i k_j)^{\frac{1}{2}} E_{ij} e_{ij} \tag{30}$$

$$L^- = \sum_{i=1}^{2n+1} k_i^{-1} e_{ii} - \sum_{i=1}^{2n} \alpha_i (k_i k_{i+1})^{-\frac{1}{2}} X_i^- e_{i+1i} - \sum_{i<j-1} \alpha_{ij} (k_i k_j)^{-\frac{1}{2}} F_{ji} e_{ji} \tag{31}$$

which is a direct generalization of the ansatz of B_2. The parameters α_i and α_{ij} are appropriate numerical factors. We also have

$$k_i' = k_i^{-1} \qquad k_{n+1} = 1\,. \tag{32}$$

Equations (12)–(18) are now generalized to

$$k_i k_j = k_j k_i \tag{33}$$

$$k_i X_j^+ = X_j^+ k_i \qquad i \neq j, j+1 \tag{34}$$

$$k_i X_i^+ = q_i^{-1} X_i^+ k_i \tag{35}$$

$$k_{i+1} X_i^+ = q_{i+1} X_i^+ k_{i+1} \tag{36}$$

$$(q_i - q_{i+1}) X_i^{+2} = 0 \qquad i \neq n \tag{37}$$

more generally

$$(q_i - q_j) L_{ij}^{+2} = 0 \qquad i \neq i' \quad j \neq j' \tag{38}$$

$$X_i^+ X_j^+ = X_j^+ X_i^+ \qquad |i-j| \geqslant 2 \qquad j \neq i' \tag{39}$$

$$X_i^+ X_{i'}^+ = \frac{q_i}{q_{i+1}} X_{i'}^+ X_i^+ \qquad i < i'\,. \tag{40}$$

It is very interesting to note that when $q_i \neq q_{i+1}$, X_i+ and $X+_{i'}$ are nilpotent and they can not be identified with each other. In the standard case (all $q_i = q$) the commutations relations of X_i^+ and $X_{i'}^+$ with all the other elements of the algebra are the same and (40) also shows that they commute with each other, so they can be identified with each other. The same relations as (33)–(40) are obtained from (6), with $X_i^+ \to X_i^-$ and $q_i \to {q_i}^{-1}$. From (7) we also derive the following:

$$[X_i^+, X_j^-] = \delta_{ij} \frac{k_{i+1} k_i^{-1} - k_i k_{i+1}^{-1}}{q^{d_i} - q^{-d_i}}\,. \tag{41}$$

To obtain the structure of $X_q(B_n)$, we note that in the general case, if we insist on preserving the form of the Cartan matrix of B_n then $k_{i+1}k_i$ can not be directly identified with qH_i. In fact this element has the following commutation relations with simple roots, which can be written compactly as

$$k_{i+1} {k_i}^{-1} X_j^+ = {q_i}^{\delta_{ij} - \delta_{i-1,j}} q_{i+1}^{\delta_{ij} - \delta_{i+1,j}} X_j^+ k_{i+1} {k_i}^{-1}\,. \tag{42}$$

Therefore we set

$$\frac{k_{i+1}}{k_i} = q^{d_i} H_i \Theta_i \tag{43}$$

where H_i's are the generators of the Cartan subalgebra $d_i = \mathrm{diag}(1, 1, \ldots, 1, \frac{1}{2})$ is the symmetrizer of the Cartan matrix of B_n and Θ_i ($i = 1, \ldots, n$) are new generators in the Cartan subgroup, which we call twist generators, and their commutation relations are determined to be

$$\Theta_i X_j^+ = \omega_{ij} X_j^+ \Theta_i \tag{44}$$

where

$$\omega_{ij} = \frac{q_i^{\delta_{ij}-\delta_{i-1,j}} q_{i+1}^{\delta_{ij}-\delta_{i+1,j}}}{q^{d_i a_{ij}}} \,. \tag{45}$$

We call ω_{ij} the twisting matrix.

It was first introduced in [7] in connection with the $X_q(A_n)$. We give the form of the twisting matrix for some simple cases:

$$w(A_2) = \begin{pmatrix} q_1 q_2/q^2 & q/q_2 \\ q/q_2 & q_2 q_3/q^2 \end{pmatrix} \tag{46}$$

$$w(A_n) = \begin{pmatrix} \omega(A_{n-1}) & & \\ & & q/q_n \\ & q/q_n & q_n q_{n+1}/q^2 \end{pmatrix} \tag{47}$$

$$w(B_n) = \begin{pmatrix} \omega(A_{n-1}) & & \\ & & q/q_n \\ & q/q_n & q_n/q \end{pmatrix} . \tag{48}$$

It is easy to find the operators Θ_i. Setting $\Theta_i = e^{\sum c_{ik} H_k}$ and $\omega_{ij} = e^{t_{ij}}$ one finds that $\sum c_{ik} a_{kj} = t_{ij}$ where $[a_{ij}]$ is the cartan matrix of B_n. Therefore

$$\Theta_i = \prod_{j,k} \omega_{ij}^{a_{jk}^{-1}} H_k \,. \tag{49}$$

The structure of $X_q(B_n)$ now becomes

$$(q_i - q_{i+1}) X_i^{\pm 2} = 0 \qquad i \neq n \tag{50}$$

$$[H_i, H_j] = 0 \tag{51}$$

$$[H_i, X_j^{\pm}] = \pm a_{ij} X_j^{\pm} \tag{52}$$

$$[X_i^{\pm}, X_j^{\pm}] = 0 \qquad \text{if} \quad a_{ij} = 0 \tag{53}$$

$$[X_i^+, X_j^-] = \delta_{ij} \frac{q^{d_i H_i} \Theta_i - q^{-d_i H_i} \Theta_i^{-1}}{q^{d_i} - q^{-d_i}} \,. \tag{54}$$

The Hopf structure obtained from the standard co-action of FRT [2] is

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \tag{55}$$

$$\Delta(X_i^{\pm}) = q^{-\frac{1}{2}d_iH_i}\Theta_i^{-\frac{1}{2}} \otimes X_i^{\pm} + X_i^{\pm} \otimes q^{\frac{1}{2}d_iH_i}\Theta_i^{\frac{1}{2}} \tag{56}$$

$$\epsilon(H_i) = \epsilon(X_i^{+}) = 0 \qquad \epsilon(1) = 1 \tag{57}$$

$$S(H_i) = -H_i \tag{58}$$

$$S(X_i^{+}) = -(q_iq_{i+1})^{\frac{i}{2}}X_i^{+}\,. \tag{59}$$

It is clear that

$$\Delta(\Theta_i) = \Theta_i \otimes \Theta_i \qquad \epsilon(\Theta_i) = 1 \qquad S(\Theta_i) = \Theta_i^{-1}\,. \tag{60}$$

It can be easily verified that with the above definitions, all the axioms of Hopf algebra are satisfied. We urge the reader to check how this co-product is compatible with the nil-potencies of the generators, i.e. $\Delta((X_i^{\pm})^2) = 0$.

Serré relations. In order to derive Serré relations as in the case of B_2, we must eliminate E_i from the following sets of relations: (derived from (6) where E_i is shorthand for E_{ii+2})

$$[X_{i+1}^{+}, E_i]_{q_{i+2}} = 0 \qquad i \neq n-1 \tag{61}$$

$$[X_i^{+}, E_i]_{q_i-1} = 0 \qquad i \neq n \tag{62}$$

$$[X_i^{+}, X_{i+1}^{+}]_{q_{i+1}} = -E_i \qquad i \neq n\,. \tag{63}$$

This elimination leads immediately to

$$q_iX_i^{+2}X_{i\pm 1}^{+} - (1 + q_iq_{i+1})X_i^{+}X_{i\pm 1}^{+}X_i^{+} + q_{i+1}X_{i\pm 1}^{+}X_i^{+2} = 0 \qquad i \neq n\,. \tag{64}$$

The quartic Serré relations of B_n is obtained by eliminating $E_{n-1,n+2}$ and $E_{n-1,n+1}$ from

$$[X_n^{+}, E_{n-1,n+2}]_{q_n} = 0 \tag{65}$$

$$[X_n^{+}, E_{n-1,n+1}] = -E_{n-1,n+2} \tag{66}$$

$$[X_{n-1}^{+}, X_n^{+}]_{q_n} = -E_{n-1,n+2}\,. \tag{67}$$

This then leads to

$$X_n^{+3}X_{n-1}^{+} - \binom{3}{1}_{q_n^{\frac{1}{2}}}X_n^{+2}X_{n-1}^{+}X_n^{+} + \binom{3}{2}_{q_n^{\frac{1}{2}}}X_n^{+}X_{n-1}^{+}X_n^{+2} + X_{n-1}^{+}X_n^{+3} = 0 \tag{68}$$

where

$$\left(n_k\right)_q = \frac{[n]_q!}{[k]_q![n-k]_q!} \qquad \text{and} \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}\,. \tag{69}$$

In the standard case (all $q_i = q$), the above Serré relations coincide with the Serré relations of $U_q(B_n)$. Now let $q_i \neq q_{i+1}$ (i.e. $q_iq_{i+1} = -1$) then from (50) we have $X_i^{\pm 2} = 0$, so the Serré relations (64) become trivial identities and provide no information as to how one must pass X_i^{+} through $X_{i\pm 1}^{+}$ in arbitrary monomials in the enveloping algebra. This is a general feature of non-standard quantum groups, in which reconstruction of the whole algebra in the Cartan–Weyl basis is impossible from the Chevally basis. This is also what happens in the case of supergroups [8]. This fact, together with nil-potency of the generators, suggests that there is a strong relation between non-standard quantum groups and the graded algebras.

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